Arbitrary-amplitude electrostatic traveling structures in a plasma

Alexander O. Harin*
Department of Mathematics and Applied Mathematics, University of Natal, Durban 4001, South Africa

Valentin O. Strasser†
Plasma Physics Research Institute, Department of Physics, University of Natal, Durban 4001, South Africa
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A fluid model of a plasma in the case of a general polytropic process is considered. An alternative method of analysis of electrostatic traveling structures to the formalism of the Sagdeev pseudopotential [Rev. Plasma Phys. **4**, 23 (1966)] is used to obtain an existence domain for compressive solitons and to establish the absence of rarefactive solitons and monotonic double layers in a two-component plasma.

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I. INTRODUCTION

The study of arbitrary-amplitude traveling structures (such as solitons, double layers, etc.) in plasmas has been a subject of considerable interest in recent years. The assumption of the arbitrariness of the structure amplitude makes the Korteweg–de Vries equation inapplicable, and one should use the more general fluid model. Starting from the work of Ref. [1], the traveling structure solutions of fluid equations were usually analyzed using the formalism of the Sagdeev pseudopotential. It was shown [2] (in the approximation of Boltzmann electrons and cold ions) that plasmas consisting of single ion and electron components do not admit rarefactive solitons, though the question of their existence in a more general case remained open. Recently there appeared many papers studying traveling structures in multicomponent plasmas [3,4]. The question of special interest was the domain of existence of such solutions. Numerical investigation [5,6] has shown the existence of considerable restrictions on the range of parameters for such solutions to be possible.

It is found from theory [7–10] and experiments [11–13] that the particle distribution within a kink (double layer) can be classified into *trapped* and *free* groups. (Hereafter by a kink we mean a monotonic transition layer.) This implies that in a fluid model of a two-component plasma kinks do not occur, though a rigorous mathematical proof of this fact has not been given.

In this article we study the existence conditions for the traveling structures without using the conventional method of the Sagdeev pseudopotential. The outline of the paper is as follows. In Sec. II we introduce the fluid model of a plasma. In Sec. III we perform a partial integration of the model using a "traveling structure" *ansatz.* This results in two constraints on the density configurations. The first one defines a line and the second defines a region in density space. In Sec. IV we restrict ourselves to a two-component plasma and formulate a necessary and sufficient condition for the existence of solitons and kinks in terms of the mutual geometrical locations of the constraints on the density plane. As a by-product of this consideration we establish the non-existence of rarefactive solitons. In Sec. V we show that a two-component plasma with the same thermodynamic properties of the components cannot support double layers. The domain of existence of compressive solitons is found in Sec. VI. Finally, Sec. VII is devoted to concluding remarks.

II. MODEL

The plasma is assumed to be infinite, homogeneous, collisionless, unmagnetized, and quasineutral. The system of plasma fluid equations is then given by

$$\frac{\partial n_j}{\partial t} + \frac{\partial (n_j v_j)}{\partial x} = 0, \tag{2.1a}$$

$$m_j n_j \left(\frac{\partial v_j}{\partial t} + v_j \frac{\partial v_j}{\partial x} \right) + \frac{\partial p_j}{\partial x} = -e_j n_j \frac{\partial \phi}{\partial x}, \tag{2.1b}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi \sum_j e_j n_j. \tag{2.1c}$$

Here $n_j, m_j, e_j, v_j,$ and p_j are the density, mass, charge, velocity, and pressure of the species j, respectively. To obtain a closed system one should add an equation of state. We assume a general polytropic process

$$\frac{p_j}{n_j^{\gamma_j}} = \text{const}, \tag{2.2}$$

where γ_j is the polytropic index.

We assume the following boundary conditions:

$$\phi, \frac{\partial \phi}{\partial x}, v_j \to 0; \quad n_j \to n_{j0}; \quad p_j \to n_{j0} T_j, \tag{2.3}$$

*Electronic address: strasser@ph.und.ac.za
†Electronic address: harin@lourie.und.ac.za

as $x \to -\infty$. Here T_j is the background temperature and the unperturbed density n_{j0} satisfies the neutrality condition

$$\sum_j e_j n_{j0} = 0. \tag{2.4}$$

III. ARBITRARY-AMPLITUDE TRAVELING STRUCTURES

We look for traveling structure solutions propagating with a constant velocity u. It is advantageous to transform to a moving frame making the transformation $\xi = x - ut$.

Equation (2.1a) can then be readily integrated, yielding

$$v_j = u\left(1 - \frac{n_{j0}}{n_j}\right).$$

Substituting this into Eq. (2.1b) we obtain

$$\left[\gamma_j T_j \left(\frac{n_j}{n_{j0}}\right)^{\gamma_j - 1} - m_j u^2 \left(\frac{n_{j0}}{n_j}\right)^2\right] \frac{\partial n_j}{\partial \xi} = -e_j n_j \frac{\partial \phi}{\partial \xi}. \tag{3.1}$$

Equation (3.1) can be integrated to give

$$T_j \ln x_j + \frac{m_j u^2}{2}\left(x_j^{-2} - 1\right) = -e_j \phi \tag{3.2a}$$

for an isothermal process ($\gamma_j = 1$) or

$$\frac{\gamma_j T_j}{\gamma_j - 1}(x_j^{\gamma_j - 1} - 1) + \frac{m_j u^2}{2}\left(x_j^{-2} - 1\right) = -e_j \phi \tag{3.2b}$$

for an anisothermal process ($\gamma_j \neq 1$), where $x_j = n_j/n_{j0}$ are the normalized densities. Eliminating ϕ from Eqs. (3.2) we obtain

$$\sum_{\gamma_j \neq 1}\left[\frac{\gamma_j T_j}{\gamma_j - 1}(x_j^{\gamma_j - 1} - 1) + \frac{m_j u^2}{2}\left(x_j^{-2} - 1\right)\right] n_{j0} + \sum_{\gamma_j = 1}\left[T_j \ln x_j + \frac{m_j u^2}{2}\left(x_j^{-2} - 1\right)\right] n_{j0} = 0. \tag{3.3}$$

This equation defines a curve in a space of normalized densities x_j, which we shall further refer to as a trajectory of the solution. We denote the left-hand side of Eq. (3.3) by $\mathcal{T}(\mathbf{x})$.

On the other hand, summing Eq. (3.1) over all types of species, using the Poisson equation (2.1c), and integrating we obtain

$$\sum_j \left[T_j(x_j^{\gamma_j} - 1) + m_j u^2 (x_j^{-1} - 1)\right] n_{j0} = \frac{1}{8\pi}\left(\frac{\partial \phi}{\partial \xi}\right)^2. \tag{3.4}$$

We denote the left-hand side of Eq. (3.4) as $\mathcal{B}(\mathbf{x})$. Since $(\partial\phi/\partial\xi)^2 \geq 0$, configurations with densities satisfying the inequality $\mathcal{B} < 0$ are not allowed. The boundary of this region of "impossibility of motion" is defined by

$$\sum_j \left[T_j(x_j^{\gamma_j} - 1) + m_j u^2 (x_j^{-1} - 1)\right] n_{j0} = 0. \tag{3.5}$$

IV. TWO-COMPONENT PLASMA

We restrict ourselves to the plasma consisting of two species $j = 1, 2$. In this case both the trajectory and the boundary are curves in the plane of the densities (x_1-x_2). Let k and l be fixed indices, $k = 1$ or 2, $l = 3 - k$. The trajectory and the boundary curves can be represented as solutions of autonomous differential equations [differentiating Eqs. (3.3) and (3.5), respectively]

$$\frac{dx_{tl}}{dx_k} = \frac{e_l}{e_k}\left[\frac{\gamma_k T_k x_k^{\gamma_k - 2} - m_k u^2 x_k^{-3}}{\gamma_l T_l x_{tl}^{\gamma_l - 2} - m_l u^2 x_{tl}^{-3}}\right] \tag{4.1}$$

and

$$\frac{dx_{bl}}{dx_k} = \frac{e_l}{e_k}\left[\frac{\gamma_k T_k x_k^{\gamma_k - 1} - m_k u^2 x_k^{-2}}{\gamma_l T_l x_{bl}^{\gamma_l - 1} - m_l u^2 x_{bl}^{-2}}\right], \tag{4.2}$$

respectively, with the initial condition $x_{tl}(0) = x_{bl}(0) = 1$.

The initial point $(1,1)$ belongs both to the trajectory and the boundary. This is also a point where they have a common tangent,

$$\left.\frac{dx_{tl}}{dx_k}\right|_{x_k=1} = \left.\frac{dx_{bl}}{dx_k}\right|_{x_k=1} = \frac{e_l}{e_k}\left[\frac{\gamma_k T_k - m_k u^2}{\gamma_l T_l - m_l u^2}\right]. \tag{4.3}$$

We denote $[m_j u^2/(\gamma_j T_j)]^{1/(\gamma_j+1)}$ by α_j. For $0 < \alpha_{k,l} < +\infty$ the curves defined by Eqs. (3.3) and (3.5) are closed. The extrema of x_j are achieved at $x_{3-j} = \alpha_{3-j}$, since here the derivatives dx_j/dx_{3-j} vanish (Fig. 1). According to Eq. (3.1), the derivatives $dx_j/d\xi$ become infinite when $x_j = \alpha_j$. Therefore, the solution exists only on the part of the trajectory falling into the quadrant bounded by the lines $x_j = \alpha_j$ and containing the initial point

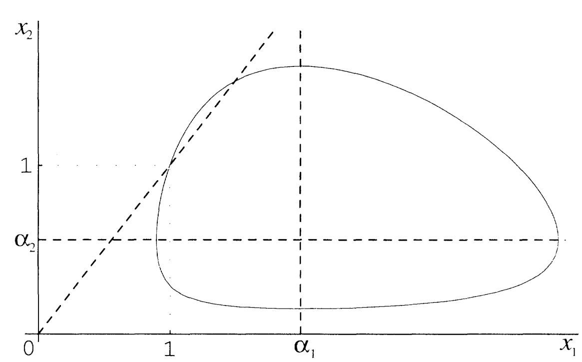

FIG. 1. Typical form of the "boundary" on the x_1-x_2 plane in the case $\alpha_2 < 1$, $\alpha_1 > 1$.

(permissible quadrant).

We shall now estimate the "escape time" from the points of the boundary. By escape time we mean the range of ξ needed for a finite change in densities x_j. Expressing $\partial\phi/\partial\xi$ from Eq. (3.4) and substituting it into Eq. (3.1) results in

$$\left(\gamma_j T_j x_j^{\gamma_j-2} - m_j u^2 x_j^{-3}\right)\frac{dx_j}{d\xi} = \mp e_j\sqrt{8\pi\sum_i n_{i0}\left[T_i(x_i^{\gamma_i}-1)+m_i u^2(x_i^{-1}-1)\right]}. \tag{4.4}$$

Let $(\varkappa_1, \varkappa_2)$ be a point of the boundary and $\varkappa_j \neq \alpha_j$. Introducing new variables $\nu_j = x_j - \varkappa_j$ and taking into account that

$$\nu_{3-j} = \nu_j \frac{e_{3-j}}{e_j}\left[\frac{\gamma_j T_j \varkappa_j^{\gamma_j-2} - m_j u^2 \varkappa_j^{-3}}{\gamma_{3-j}T_{3-j}\varkappa_{3-j}^{\gamma_{3-j}-2} - m_{3-j}u^2\varkappa_{3-j}^{-3}}\right] + O(\nu_j^2),$$

due to Eq. (4.1), we can write Eq. (4.4) as

$$\left(\gamma_j T_j \varkappa_j^{\gamma_j-2} - m_j u^2 \varkappa_j^{-3}\right)\frac{d\nu_j}{d\xi} = \mp e_j[8\pi n_{j0}\nu_j(\varkappa_{3-j}-\varkappa_j) \times (m_j u^2 \varkappa_j^{-3} - T_j\gamma_j\varkappa_j^{\gamma_j-2}) + O(\nu_j^2)]^{1/2}.$$

We observe that for $\varkappa_1 = \varkappa_2$,

$$\frac{d\nu_j}{d\xi} = \pm a_j \nu_j + O(\nu_j^2), \quad a_j > 0, \tag{4.5}$$

and for $\varkappa_1 \neq \varkappa_2$,

$$\frac{d\nu_j}{d\xi} = \pm b_j\sqrt{|\nu_j|} + O(|\nu_j|^{3/2}). \tag{4.6}$$

Equation (4.5) implies that ν_j behaves like an exponential function. It vanishes when $\xi \to \infty$, hence the trajectory may leave or enter points with $\varkappa_1 = \varkappa_2$ in an infinite "time." On the other hand, Eq. (4.6) yields a finite time of entry (escape) for the points with $\varkappa_1 \neq \varkappa_2$. Assuming $x_1 = x_2$ in Eq. (3.5), one can verify that the boundary and the bisector $x_1 = x_2$ can have at most two common points. The infiniteness of the escape time for the point $(1,1)$ proves the consistency of the boundary conditions (2.3).

At positive infinity the solution can have two different types of behavior. It can either end at the second point with the infinite escape time ($x_1 = x_2 \neq 1$) or reach a turning point on the boundary where $x_1 \neq x_2$ [at this point the sign of the right-hand side of Eq. (4.4) changes] and return to the point $(1,1)$. In the first case the solution will have the form of a kink, while the second case corresponds to a symmetric (with respect to ξ) solitary structure (soliton). In both cases the solution satisfying boundary conditions (2.3) exists if and only if the following two conditions hold: (i) the trajectory touches the region $\mathcal{B} < 0$ at the initial point (1,1) on the outer side (the escape condition) and (ii) the trajectory intersects (touches at $x_1 = x_2 \neq 1$ in the case of a kink) the boundary in the permissible quadrant. The escape condition can be written in terms of second derivatives at the initial point,

$$\left.\frac{d^2 x_{tl}}{dx_k^2}\right|_{x_k=1} < \left.\frac{d^2 x_{bl}}{dx_k^2}\right|_{x_k=1} \quad \text{if} \quad \alpha_l > 1, \tag{4.7a}$$

$$\left.\frac{d^2 x_{tl}}{dx_k^2}\right|_{x_k=1} > \left.\frac{d^2 x_{bl}}{dx_k^2}\right|_{x_k=1} \quad \text{if} \quad \alpha_l < 1. \tag{4.7b}$$

In view of Eqs. (4.1)–(4.3) these conditions can be reduced to

$$0 < \left.\frac{dx_{tl}}{dx_k}\right|_{x_k=1} = \left.\frac{dx_{bl}}{dx_k}\right|_{x_k=1} < 1 \quad \text{for} \quad \alpha_l > 1, \alpha_k < 1. \tag{4.8}$$

In the case $\alpha_1, \alpha_2 > 1$ the escape condition can never be satisfied, while in the case $\alpha_1, \alpha_2 < 1$ it always holds.

Let us suppose that the escape condition is satisfied, i.e., in the neighborhood of the initial point the trajectory is in the region $\mathcal{B} > 0$. The requirement of intersection will then be fulfilled if the trajectory is in the region $\mathcal{B} < 0$ when it reaches one of the lines $x_j = \alpha_j$ (the boundaries of the permissible quadrant). To show that this condition is also necessary we will prove that the trajectory can intersect the boundary in the permissible quadrant only once.

Let

$$f(x_1) = \frac{d(x_{t2} - x_{b2})}{dx_1} = \frac{e_2}{e_1}\frac{\gamma_1 T_1 x_1^{\gamma_1-2} - m_1 u^2 x_1^{-3}}{\gamma_2 T_2 x_{t2}^{\gamma_2-2} - m_2 u^2 x_{t2}^{-3}} - \frac{e_2}{e_1}\frac{\gamma_1 T_1 x_1^{\gamma_1-1} - m_1 u^2 x_1^{-2}}{\gamma_2 T_2 x_{b2}^{\gamma_2-1} - m_2 u^2 x_{b2}^{-2}}. \tag{4.9}$$

$f(x_1)$ is the derivative of the difference $g(x_1) = x_{t2}(x_1) - x_{b2}(x_1)$, which is a single-valued function of x_1 in the permissible quadrant. Note that $f(1) = 0$. Assuming that escape conditions (4.7a) and (4.7b) hold, we obtain that

$$\operatorname{sgn} f(1+\delta) = \operatorname{sgn}(\delta)\,\operatorname{sgn}(1-\alpha_2) \tag{4.10}$$

for sufficiently small δ. On the other hand, at a common point of the boundary and trajectory where $x_{t2} = x_{b2} = x_2$,

$$f(x_1) = \frac{e_2}{e_1}\frac{\gamma_1 T_1 x_2^3}{\gamma_2 T_2 x_1^3}\frac{x_1^{\gamma_1+1} - \alpha_1^{\gamma_1+1}}{x_2^{\gamma_2+1} - \alpha_2^{\gamma_2+1}}\left(1 - \frac{x_1}{x_2}\right). \tag{4.11}$$

We first show that in the case $\alpha_2, \alpha_1 < 1$ the trajectory and the boundary have no points of intersection in the permissible quadrant, which is now defined by $(x_1 > \alpha_1, x_2 > \alpha_2)$. Let (x_1^*, x_2^*) be the point of intersection of the boundary and the trajectory immediately to the right of the initial point $x_2^* < 1 < x_1^*$. Equation (4.10) then implies that $f(1+0) > 0$, i.e., $g(x_1)$ emerges from zero while increasing and it has to decrease when it reaches its next zero, and so $f(x_1^*) < 0$. Hence, according to Eq. (4.11), $x_2^* > x_1^*$, which contradicts the assumption made above. The same argument can be applied if the point of intersection (x_1^*, x_2^*) is assumed to be immediately to the left of the initial point.

Let us now analyze the case $\alpha_2 < 1$, $\alpha_1 > 1$ (Fig. 1). Again let (x_1^*, x_2^*) be the point of intersection immediately to the right of the initial point $x_1^* > 1$. Again Eq. (4.10) implies that $f(1+0) > 0$ and so $f(x_1^*) < 0$. However, this time Eq. (4.11) yields $x_2^* < x_1^*$. Note that condition (4.8) implies that $dx_2/dx_1 > 1$ and hence the bisector $x_1 = x_2$ intersects the boundary between $x_1 = 1$ and $x_1 = x_1^*$. Suppose there exists another point of intersection (x_1^{**}, x_2^{**}) to the right of (x_1^*, x_2^*), $x_1^{**} > x_1^*$. Then $g(x_1)$ reaches its zero at x_1^{**} while increasing, i.e., $f(x_1^{**}) > 0$, and consequently $x_2^{**} > x_1^{**}$. This means that the line $x_1 = x_2$ should again intersect the boundary, which is impossible.

If (x_1^*, x_2^*) is the point of intersection immediately to the left $x_1^* < 1$, then, since $f(1-0) < 0$, we have $f(x_1^*) > 0$. Hence $x_2^* > x_1^*$. This, however, contradicts the condition $dx_2/dx_1 > 1$, which implies that this point lies below the bisector, i.e., $x_2^* < x_1^*$.

The case $\alpha_2 > 1, \alpha_1 < 1$ can be obtained from the previous case by the formal change of indices. As a result the trajectory and the boundary can have only one point of intersection in the permissible quadrant and only in the case when $\alpha_l < 1$, $\alpha_k > 1$. Moreover, if (x_1^*, x_2^*) is such a point of intersection, $x_{1,2}^* > 1$. In this case the sufficient condition of intersection, which now becomes also necessary, can be expressed as the inequality

$$x_{tl}(\alpha_k) < x_{bl}(\alpha_k). \tag{4.12}$$

Note that the condition $x_{1,2}^* > 1$ means that densities can only increase, i.e., rarefactive solitons are not allowed.

V. AN EXCLUSION OF MONOTONIC DOUBLE LAYERS

Apart from solitons, the analysis of Sec. IV gave us another possible type of traveling structure, viz., a kink. Such a solution would exist if and only if the second common point of the boundary and trajectory in the permissible quadrant belongs also to the line $x_1 = x_2$. We shall further assume $\gamma_1 = \gamma_2 = \gamma$. Assuming $x_1 = x_2 = x$ in Eqs. (3.3) and (3.5) for the trajectory and the boundary results in

$$x^{\gamma-1} - 1 + \frac{\gamma-1}{\gamma} a \left(x^{-2} - 1\right) = 0 \tag{5.1a}$$

and

$$x^{\gamma} - 1 + a\left(x^{-1} - 1\right) = 0, \tag{5.1b}$$

where

$$a = \frac{m_1 n_{10} + m_2 n_{20}}{T_1 n_{10} + T_2 n_{20}} u^2.$$

A kink can exist only for such γ and a that a system of equations (5.1a) and (5.1b) has a solution $x \neq 1$. Multiplying Eq. (5.1a) by x^2 and subtracting it from Eq. (5.1b), multiplied by x, results in

$$x^2\left(1 + \frac{\gamma-1}{2\gamma} a\right) - x(a+1) + \frac{\gamma+1}{2\gamma} a = 0. \tag{5.2}$$

The roots of this quadratic equation are

$$x_1 = 1, \quad x_2 = \frac{(\gamma+1)\,a}{2\gamma + (\gamma-1)\,a}.$$

x_2 solves the system (5.1) if and only if it solves Eq. (5.1b). Expressing a through x_2,

$$a = \frac{2x_2\gamma}{(\gamma+1) - x_2(\gamma-1)},$$

we rewrite Eq. (5.1b) as

$$g(x_2) \stackrel{\text{def}}{=} x_2^{\gamma}\left(x_2 - \frac{\gamma+1}{\gamma-1}\right) - 1 + x_2 \frac{\gamma+1}{\gamma-1} = 0. \tag{5.3}$$

Since $g(1) = 0$ while $g'(x_2) \neq 0$ for $x_2 \neq 1$, $x_2 = 1$ is the only root in accordance with Rolle's theorem. Consequently $x = 1$ is the only common root of Eqs. (5.1a) and (5.1b).

For an isothermal process ($\gamma = 1$), instead of Eqs. (5.1a) and (5.1b), we have

$$\ln x + \frac{a}{2}\left(x^{-2} - 1\right) = 0, \tag{5.4a}$$

$$x - 1 + a\left(x^{-1} - 1\right) = 0. \tag{5.4b}$$

Equation (5.4b) has two roots $x_1 = 1$ and $x_2 = a$. Substituting $x = a$ into Eq. (5.4a) we obtain

$$\ln a = \frac{1}{2}\left(a - \frac{1}{a}\right). \tag{5.5}$$

Using the same consideration as for Eq. (5.3) we find that $a = 1$ is the only root of Eq. (5.5). This proves that the model in question cannot support monotonic transition layers.

VI. EXISTENCE DOMAIN FOR COMPRESSIVE ION-ACOUSTIC SOLITONS

Let us call a j type of particle in the plasma ions (i) if $\alpha_j > 1$ or electrons (e) if $\alpha_j < 1$. A condition necessary and sufficient for the existence of compressive solitons is given by Eqs. (4.8) and (4.12). We introduce the dimensionless parameters $\mu = m_e/m_i$, $\eta = n_{i0}/n_{e0}$, $\tau = T_i/T_e$, $\theta = u^2/V_s^2 = m_i u^2/T_e$, where V_s is ion acoustic speed. Since $\alpha_e = (\theta\mu/\gamma_e)^{1/(\gamma_e+1)} < 1$ and $\alpha_i = [\theta/(\gamma_i\tau)]^{1/(\gamma_i+1)} > 1$, θ and τ satisfy $\tau < \theta/\gamma_i$ and $\theta < \gamma_e/\mu$.

The escape condition (4.8) written in our dimensionless parameters is

$$\tau < \frac{\theta}{\gamma_i}\left(1 + \frac{\mu}{\eta}\right) - \frac{\gamma_e}{\gamma_i \eta}\,, \qquad \theta < \frac{\gamma_e}{\mu}\,. \tag{6.1}$$

Equation (6.1) gives the left margin of the existence domain (the thick straight line in Fig. 2).

The boundary of the region defined by Eq. (4.12) is given by

$$x_{te}(\alpha_i) = x_{be}(\alpha_i), \qquad (6.2)$$

where $x_{be}(\alpha_i)$ and $x_{te}(\alpha_i)$ are the greater roots of equations $\mathcal{T}(x_{te}, \alpha_i) = 0$ and $\mathcal{B}(x_{be}, \alpha_i) = 0$ respectively. (We should choose the greater roots, since we consider only those parts of the trajectory and the boundary that fall into the permissible quadrant.) Hence Eq. (6.2) is a compatibility condition of these equations.

Ion-acoustic solitons are stable and propagate with a constant speed only if $\tau \ll 1$; otherwise their amplitude and hence speed decrease due to Landau damping. Then $V_{T_e} \gg u$ ($V_{T_j} = \sqrt{2T_j/m_j}$ is the thermal velocity of particle species j) and hence the variations in the electron temperature are very small. Accordingly, the polytropic index for electrons is very close to that of an isothermal process $\gamma_e = 1$. The ion-acoustic velocity, on the other hand, although less, can be of the same order as a structure speed. Hence the process for ions can be naturally approximated as an adiabatic.

For an isothermal electron, $\gamma_e = 1$ and an anisothermal ion $\gamma_i = \gamma \neq 0$, $x_{be}(\alpha_i)$ and $x_{te}(\alpha_i)$ are found from

$$\mathcal{T}(x_{te}, \alpha_i) = \ln x_e + \frac{\theta\mu}{2}\left(x_e^{-2} - 1\right) + \frac{\gamma\tau\eta}{\gamma - 1}\left(\alpha_i^{\gamma-1} - 1\right) + \frac{\theta\eta}{2}\left(\alpha_i^{-2} - 1\right) = 0 \qquad (6.3)$$

and

$$\mathcal{B}(x_{be}, \alpha_i) = x_e - 1 + \theta\mu\left(x_e^{-1} - 1\right) + \tau\eta\left(\alpha_i^{\gamma} - 1\right) + \theta\eta\left(\alpha_i^{-1} - 1\right) = 0, \qquad (6.4)$$

according to Eqs. (3.3) and (3.5). Equation (6.4) produces a quadratic equation

$$x_e^2 - x_e b + c = 0, \qquad (6.5)$$

where $b = 1 + \mu\theta + \theta\eta\left(1 - \alpha_i^{-1}\right) + \tau\eta\left(1 - \alpha_i^{\gamma}\right) > 0$, $c = \theta\mu$.

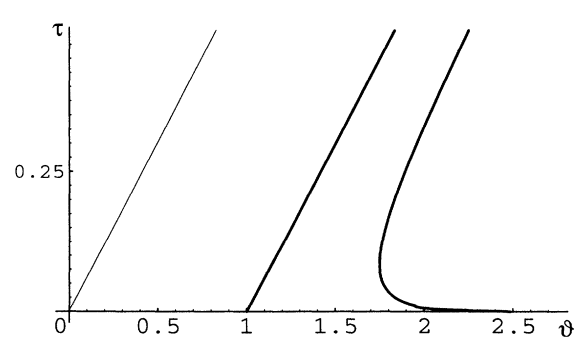

FIG. 2. Domain of existence of compressive ion-acoustic solitons for an electron-proton plasma on the τ-ϑ plane, $\tau = T_i/T_e$, $\vartheta = m_i u^2/T_e = u^2/V_s^2$. The domain is bounded by the thick lines.

To prove the positiveness of b it suffices to show that

$$\theta(1 - \alpha_i^{-1}) + \tau(1 - \alpha_i^{\gamma}) \geq 0, \qquad (6.6)$$

where $\alpha_i = [\theta/(\tau\gamma)]^{1/(\gamma+1)}$. Introducing a variable $q = \theta/\tau$, Eq. (6.6) becomes

$$\left[1 + q - q^{\frac{\gamma}{\gamma+1}}\left(\gamma^{\frac{1}{\gamma+1}} + \gamma^{-\frac{\gamma}{\gamma+1}}\right)\right]\tau = p(q)\tau \geq 0. \qquad (6.7)$$

Equation (6.7) always holds, since at $q = \gamma$, $p(q)$ achieves its minimum $p(\gamma) = 0$.

Hence the greater root of Eq. (6.5) is $x_+ = b + \sqrt{b^2 - 4ac}/2a$. Accordingly, Eq. (6.2) is equivalent to

$$\ln x_+ + \frac{\theta\mu}{2}\left(x_+^{-2} - 1\right) + \frac{\gamma\tau\eta}{\gamma - 1}\left(\alpha_i^{\gamma-1} - 1\right) + \frac{\theta\eta}{2}\left(\alpha_i^{-2} - 1\right) = 0\ . \qquad (6.8)$$

Equation (6.8), as one can verify, is satisfied on a line segment $\tau = \theta/\gamma$, $0 < \theta \leq \gamma/\mu$ (thin straight line in Fig. 2). A numerical solution shows that Eq. (6.8) is also satisfied on another line that gives the right margin of the existence domain (Fig. 2) (thick curve in Fig. 2).

VII. CONCLUDING REMARKS

We have obtained the following important results for a fluid model of a two-component plasma.

(1) The model does not admit monotonic transition layers in the case of a plasma consisting of two types of particles which have the same thermodynamic properties ($\gamma_1 = \gamma_2$).

(2) The model does not admit rarefactive solitons.

(3) Compressive ion-acoustic solitons exist only in the domain defined by Eqs. (6.1) and (6.8).

In Fig. 2 we present the domain of existence of compressive solitons in a plasma consisting of a single isothermal electron species and a single adiabatic proton species, $\mu = 1/1836$, $\gamma_e = 1$, $\gamma_i = 5/3$. These solitons are supersonic, $\vartheta \geq 1$. Let us note that the right boundary of the domain for $\tau = 0$ (cold ions) corresponds to the speed $u \approx 1.58V_s$, which coincides with the standard result of Sagdeev [1].

For low ion temperature the domain of existence sharply shrinks with the temperature growth. Consequently, even comparatively low ion temperature cannot be neglected.

In this paper we assumed that the polytropic indices are constants and do not depend on other parameters. This is in fact a simplification of the reality, since polytropic indices do depend on parameters such as the ratio of the structure speed and the thermal velocity of particles, the structure amplitude, etc. The natural way to deal with this would be to use kinetic theory, where there is no need to make assumptions about the dependence of polytropic indices. Although this approach is more physical, it seems to be much too difficult to obtain even the most simple, general conclusions about properties of traveling structures. Also there seem to be no visible possibilities of obtaining useful analytical results using the kinetic approach.

We did not consider solutions with trajectories intersecting the lines $x_j = \alpha_j$. However, we note that such solutions may be possible and would correspond to shock waves. The analysis of shock waves requires the use of the appropriate integral relations and lies beyond the scope of the present work.

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